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# Letter to the Editors

## Use of Richardson Extrapolation for the Numerical Calculation of Fourier Transforms

In the paper by Abramovici [1] a "trapezoidal Fast Fourier Transform" is introduced by removing the linear trend. In the final section of [1] the Richardson extrapolation method was used to improve the accuracy obtained by the "trapezoidal FFT", this scheme was called the Romberg procedure. The use of the linear trend has been discussed by Ng [2] and Abramovici [3]. These papers also included the corrections to the Filon values of [1].

In the works by Abramovici, [1, 3], one gets the impression that both the use of the FFT to speed up the calculation of the sum $\sum f(t_j) \exp(-i\omega t_j)$ and the use of Richardson extrapolation to improve the accuracy are restricted to the "trapezoidal FFT". However, using the Euler–Maclaurin summation formula it is easy to show for sufficiently smooth functions that the error expansion of the Filon formula contains only even powers of $h$, starting with the fourth. This behavior is most useful if $|\omega h| < \pi/2$, where $\omega$ is the angular frequency in the Fourier transform

$$\int_a^b f(t)\, e^{-i\omega t}.$$

In the proof we start with the Euler–Maclaurin summation formula

$$h \cdot \left[\frac{1}{2} g_0 + \sum_{j=1}^{N-1} g_j + \frac{1}{2} g_N\right] = \int_a^b g(t)\, dt$$

$$+ \frac{h^2}{12}\{g'(b) - g'(a)\} - \frac{h^4}{720}\{g'''(b) - g'''(a)\} + \cdots,$$

where $h = (b - a)/N$ and $g_j = g(a + jh)$.

We use $g(t) = f(t) \cdot \cos \omega t$ in this summation formula to obtain expressions for the two sums

$$\sum_{j=0}^{p} f_{2j} \cos \omega t_{2j} \qquad \text{and} \qquad \sum_{j=1}^{p} f_{2j-1} \cos \omega t_{2j-1}$$

in the Filon cosine formula. Substituting these expressions, which only contain the requested integral and endpoint corrections, into the Filon formula and also utilizing series expansions of the Filon parameters in the stepsize $h$ completes the proof for the cosine integral. The same procedure is performed for the sine integral.

Because of the simple nature of the power series expansion of the error term, it is therefore possible to use Richardson extrapolation also for the Filon method. The sums may be calculated with the Fast Fourier Transform technique, provided that the interval and frequency correspond in the right manner.

For a method using cubic splines [4] instead of the quadratic in Filon's formula, the extrapolation above has been proved in [5] and used in [6]. The calculations of Abramovici [1] have been repeated here with both the Filon and spline methods, including the use of Richardson extrapolation. These calculations, performed on an IBM 360/75 in double precision, show that the spline method is superiour to the other methods, provided that the required first- and second-order derivatives at the end points are available. The used routines are simple revisions of the Filon algorithm by Chase and Fosdick [7] and the spline algorithm [6] by the present author.

1. *Calculations without extrapolation.* In Table I the results with the two

TABLE I

Values of $\Phi_I(\omega) = \int_0^{2\pi} f(t) \sin \omega t \, dt$ for $f(t) = t \cos \omega_0 t,\ \omega_0 = 1$[a]

| $\omega$ | No. of integration points | Filon | Spline | Additional number of correct digits |
|---|---|---|---|---|
| 1 | 65 | $-1.5708$ | $-1.57080$ | 1 |
| | 257 | $-1.570796$ | $-1.57079633$ | 2 |
| | 1025 | $-1.570796327$ | $-1.5707963268$ | 1 |
| | 2049 | $-1.5707963268$ | $-1.5707963268$ | 0 |
| 30 | 65 | $-2.097 \times 10^{-1}$ | $-2.09672 \times 10^{-1}$ | 2 |
| | 257 | $-2.0967248 \times 10^{-1}$ | $-2.09672480 \times 10^{-1}$ | 1 |
| | 1025 | $-2.0967247966 \times 10^{-1}$ | $-2.09672479661 \times 10^{-1}$ | 1 |
| | 2049 | $-2.09672479661 \times 10^{-1}$ | $-2.09672479661 \times 10^{-1}$ | 0 |
| 100 | 257 | $-6.283814 \times 10^{-2}$ | $-6.2838137 \times 10^{-2}$ | 1 |
| | 1025 | $-6.28381369 \times 10^{-2}$ | $-6.2838136885 \times 10^{-2}$ | 2 |
| | 2049 | $-6.283813689 \times 10^{-2}$ | $-6.283813688548 \times 10^{-2}$ | 3 |
| 500 | 1025 | $-1.2566421 \times 10^{-2}$ | $-1.256642088 \times 10^{-2}$ | 2 |
| | 2049 | $-1.25664208800 \times 10^{-2}$ | $-1.256642088004 \times 10^{-2}$ | 1 |

[a] Accurate to all digits shown.

TABLE II

Values of $\Phi_I(\omega) = \int_0^{2\pi} f(t) \sin \omega t \, dt, \quad \text{for} \quad f(t) = t \cos \omega_0 t, \ \omega_0 = 50$[a]

| $\omega$ | No. of integration points | Original Filon values | Extrapolated values | | | |
|---|---|---|---|---|---|---|
| 1 | 257 | 2.38303851736 | | | | |
| | 513 | 2.50787059663 | 2.51619273525 | | | |
| | 1025 | 2.51390156248 | 2.51430362687 | 2.51427364102 | | |
| | 2049 | 2.51425652328 | 2.51428018733 | 2.51427981528 | 2.51427983949 | |
| | 4097 | 2.51427838293 | 2.51427984024 | 2.51427983473 | 2.51427983481 | $2.51427983481 \times 10^{-2}$ |
| 30 | 257 | 1.12196868711 | | | | |
| | 513 | 1.17520512458 | 1.17875422042 | | | |
| | 1025 | 1.17792176875 | 1.17810287836 | 1.17809253960 | | |
| | 2049 | 1.17808634990 | 1.17809732198 | 1.17809723378 | 1.17809725219 | |
| | 4097 | 1.17809656524 | 1.17809724626 | 1.17809724506 | 1.17809724510 | $1.17809724509 \times 10^{-1}$ |
| 100 | 257 | −7.03614192904 | | | | |
| | 513 | −8.36854079155 | −8.45736738238 | | | |
| | 1025 | −8.37646980462 | −8.37699840549 | −8.37572270745 | | |
| | 2049 | −8.37750496440 | −8.37757397505 | −8.37758311108 | −8.37759040680 | |
| | 4097 | −8.37757560635 | −8.37758031581 | −8.37758041646 | −8.37758040589 | $-8.37758039611 \times 10^{-2}$ |

[a] Filon's method with Richardson extrapolation. Erroneous digits italic.

TABLE III

Values of $\Phi_I(\omega) = \int_0^{2\pi} f(t) \sin \omega t$, for $f(t) = t \cos \omega_0 t$, $\omega_0 = 50$[a]

| $\omega$ | No. of integration points | Original spline values | Extrapolated values | | | |
|---|---|---|---|---|---|---|
| 1 | 257 | 2.50785693681 | | | | |
| | 513 | 2.51380729581 | 2.51420398641 | | | |
| | 1025 | 2.51424924501 | 2.51427870829 | 2.51427989435 | | |
| | 2049 | 2.51427790665 | 2.51427981742 | 2.51427983503 | 2.51427983479 | |
| | 4097 | 2.51427971404 | 2.51427983454 | 2.51427983481 | 2.51427983481 | $2.51427983481 \times 10^{-2}$ |
| 30 | 257 | 1.17438164241 | | | | |
| | 513 | 1.17786464627 | 1.17809684653 | | | |
| | 1025 | 1.17808273674 | 1.17809727610 | 1.17809728292 | | |
| | 2049 | 1.17809633890 | 1.17809724571 | 1.17809724522 | 1.17809724508 | |
| | 4097 | 1.17809718847 | 1.17809724511 | 1.17809724510 | 1.17809724510 | $1.17809724510 \times 10^{-1}$ |
| 100 | 257 | −8.25778332875 | | | | |
| | 513 | −8.37497192725 | −8.38278450048 | | | |
| | 1025 | −8.37746408554 | −8.37763022943 | −8.37754841560 | | |
| | 2049 | −8.37757376632 | −8.37758107837 | −8.37758029820 | −8.37758042323 | |
| | 4097 | −8.37758000380 | −8.37758041963 | −8.37758040917 | −8.37758040961 | $-8.37758040959 \times 10^{-2}$ |

[a] Spline method with Richardson extrapolation. Erroneous digits italic.

methods *without* extrapolation are given. The entries are the correctly rounded values with so many digits that agree with the corresponding exact value. The table is to be compared with [1, Table I] of Abramovici. We see that the spline method is about one digit more accurate than the correct Filon formula [2, Table I] and much more accurate than the other methods.

2. *Calculations with Richardson extrapolation.* In Tables II and III the results with the two methods with use of Richardson extrapolation (Romberg procedure) are given. Because of the extrapolation, all entries are given with 12 digits, with those in error in italic. These tables are to be compared with [1, Table IV] of Abramovici, where the Romberg procedure is used on the "trapezoidal FFT." We see, that although the original values from the spline method are more accurate than those of the trapezoidal FFT, the difference in accuracy is smaller in the extrapolated values.

It is important to note that the initial subdivision of the interval $[0, 2\pi]$ into 256 subintervals gives $\theta = \omega h = \omega \cdot (2\pi/256)$, which is less than $\pi/2$ up until $\omega = 64$, while the largest value of $\omega$ is 100. The special feature of the Filon and spline methods, consisting of dividing the integrand into one polynomial part and one trigonometric part, is not so essential when many subdivisions of the basic interval of length $\omega h = \pi/2$ have been done. The best performance with these methods is when the basic interval is used, and the polynomial part is nonoscillatory.

*Other methods.* Silliman [8] has discussed higher-order spline methods for the evaluation of Fourier transforms, showing that the quintic approximation gives a considerably better result than the cubic approximation. Other approaches to the use of spline approximation have been given by Ostrander [9], and Gaissmaier [10].

If the function to be transformed is available at all points and not only at an equidistant partition of the interval, the Gaussian method of Piessens *et al.* [11, 12], seems to be the most efficient one.

Professor R. Piessens has kindly pointed out to me the report by Jacobs [13] on Richardson extrapolation for the Filon–trapezoidal rule of Tuck [14]. This report contains both a strict mathematical theory and an Algol procedure for a modified extrapolation method, with improved convergence compared with the usual Richardson extrapolation method.

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